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**ON THE ANALYSIS OF THERMAL STRESSES IN VISCOELASTIC SOLIDS**

**by**

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## On the analysis of thermal stresses in viscoelastic solids\*

by

Eli Sternberg

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1. Introduction.

The study of thermal stresses and deformations in viscoelastic solids has, for various technological reasons, attracted growing attention during the past few years. This paper, which is devoted exclusively to solids that under isothermal conditions and for infinitesimal strains exhibit linear viscoelastic behavior, is intended to serve a dual purpose. First, an attempt is made to sketch a systematic account of relevant recent theoretical developments; second, we hope to supply a helpful, if necessarily incomplete, guide to available specific results, and to the literature on the subject at hand.

The present treatment is largely confined to the quasi-static analysis of thermal stresses in homogeneous and isotropic, linear viscoelastic media. Questions regarding inhomogeneous and anisotropic solids, inertia effects, and thermo-mechanical coupling effects, are touched upon only at the end of the paper (Section 7).

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The first part of the paper (Sections 2,3,4) deals with thermo-viscoelasticity theory within the assumption that the mechanical response of the material is temperature-independent. Since the rate processes of viscoelasticity are known to be highly sensitive to temperature changes, this assumption is remote from physical reality. The theory of temperature-independent materials would nevertheless appear to merit attention as a useful preliminary to a more realistic treatment of the problem and because it has in fact been taken as the basis of numerous particular investigations.

Subsequent portions of the paper (Sections 5,6) deal with temperature-dependent viscoelastic solids, particular emphasis being placed on the theory of thermo-rheologically simple materials. This theory rests on the hypothesis that a constant temperature change of the entire solid affects its mechanical response merely within a uniform distortion of the time scale. It should be apparent that the smaller amount of space devoted here to the influence of temperature-dependent response characteristics does not reflect the relative importance of this influence but rather the increased complexity of the problem once the temperature-dependence of the material is taken into account.

Although throughout the following discussions some attention is given to stress-strain relations in differential-equation form, our primary preoccupation is with the more general integral laws appropriate to solids with a continuous spectrum of relaxation or retardation times. Such a preference is motivated

by the well-known fact that the finite models governed by the differential operator law (despite their heuristic value and traditional popularity) provide an inadequate basis for the description and prediction of actual viscoelastic behavior over any appreciable range of time or frequency. This shortcoming is even more pronounced in temperature-dependent models.

The following treatment of thermo-viscoelasticity theory is strongly influenced by, and draws heavily on the results obtained in, an earlier study of the isothermal theory contained in Reference [1]. In particular, we make extensive use of the properties of Stieltjes convolutions established in [1], which enable us to cope economically with field histories that are discontinuous in time. As emphasized in [1], the algebra and calculus of convolutions furnish the natural basic tool of linear viscoelasticity theory. In contrast, an excessive reliance on integral transforms has occasionally tended to obscure the issues under consideration. Some of the theoretical results included in this paper, though elementary in character, seem to be unavailable elsewhere. In this sense the present paper is not wholly expository.

## 2. Temperature-independent mechanical response: formulation of boundary-value problems.

The fundamental system of field equations governing the quasi-static linear theory of viscoelastic solids, in the absence of thermo-mechanical coupling effects, consists of the linearized

displacement-strain relations, the stress equations of equilibrium, and the appropriate linear hereditary stress-strain law. We refer this system of equations to rectangular cartesian coordinates  $x_i$  and adopt the usual indicial notation.<sup>2</sup>

Let  $u_i(\underline{x}, t)$ ,  $\epsilon_{ij}(\underline{x}, t)$ ,  $\sigma_{ij}(\underline{x}, t)$  be the component values of the displacement, infinitesimal strain, and stress field histories at a material point with the position vector<sup>3</sup>  $\underline{x}$  and at the time  $t$ . The displacement-strain relations then become

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (2.1)$$

whereas the stress equations of equilibrium assume the form

$$\sigma_{ij,j} + F_i = 0, \quad \sigma_{ji} = \sigma_{ij}, \quad (2.2)$$

provided  $F_i$  denotes the components of the body-force density field history.

With a view toward a convenient statement of the relevant constitutive relations for isotropic viscoelastic solids we introduce the deviatoric components of strain and stress by means of

$$e_{ij} = \epsilon_{ij} - \frac{1}{3} \delta_{ij} \epsilon_{kk}, \quad s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}, \quad (2.3)$$

in which  $\delta_{ij}$  is the Kronecker delta. Next, we designate by  $T(\underline{x}, t)$

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<sup>2</sup>Latin subscripts, unless otherwise specified, have the range of the integers (1,2,3); summation over repeated subscripts is implied and subscripts preceded by a comma indicate partial differentiation with respect to the corresponding cartesian coordinate.

<sup>3</sup>Underlined letters designate vectors.

the local instantaneous temperature and let  $T_0$  be an arbitrarily chosen reference temperature. We then call the function  $\Theta$  defined by

$$\Theta = T - T_0 \quad (2.4)$$

the temperature field history. Finally, we write  $\alpha$  for the coefficient of thermal expansion, which we shall suppose to be constant. If instead  $\alpha$  is a function of the temperature, all of the succeeding considerations undergo an entirely elementary generalization: in this instance one need merely replace  $\alpha$  by  $\alpha_0 = \alpha(T_0)$  and (2.4) by

$$\Theta(\underline{x}, t) = \frac{1}{\alpha_0} \int_{T_0}^{T(\underline{x}, t)} \alpha(T') dT'. \quad (2.5)$$

The appropriate linear isotropic hereditary stress-strain relations in the form of the relaxation integral law now appear as

$$s_{ij} = e_{ij} * dG_1, \quad \sigma_{kk} = (\epsilon_{kk} - 3\alpha\Theta) * dG_2. \quad (2.6)$$

Here  $G_1$  and  $G_2$  designate the respective relaxation moduli in shear and isotropic compression. These moduli are at present functions of the time exclusively since we confine our attention to homogeneous solids and assume until later on that the material's mechanical response is temperature-independent. In writing (2.6) we have employed a notation for Stieltjes convolutions introduced previously [1] in connection with the isothermal theory. Thus, if  $f$  and  $g$  are functions of position and time,  $\omega = f * dg$  stands for the function defined by the Stieltjes integral

$$\omega(\underline{x}, t) = [f * dg](\underline{x}, t) = \int_{t'=-\infty}^t f(\underline{x}, t-t') dg(\underline{x}, t'), \quad (2.7)$$

provided this integral is meaningful.

We observe that according to (2.6), the local stress tensor at each fixed instant is a linear, continuous, and isotropic functional of the entire preceding local strain and temperature histories; moreover, this mapping of the strain and temperature histories into an associated history of stress is invariant under a translation of the time scale and has the property that the stresses induced by a free thermal expansion vanish identically. One can show<sup>4</sup> (under suitable regularity assumptions) that (2.6) is the most general stress-strain law conforming to the foregoing requirements.

The field equations (2.1), (2.2), (2.6) must hold throughout the space-time domain<sup>5</sup>  $\mathcal{R} \times (-\infty, \infty)$ , i.e. for all  $(\underline{x}, t)$  such that  $\underline{x}$  is in the open region  $\mathcal{R}$  occupied by the interior of the body and  $t$  lies in the interval  $(-\infty, \infty)$ . To these field equations we may, without essential loss of generality, adjoin the requirement that the body is originally undisturbed in the sense of the initial conditions

$$u_i = \varepsilon_{ij} = \sigma_{ij} = F_i = \Theta = 0 \text{ on } \mathcal{R} \times (-\infty, 0), \quad (2.8)$$

<sup>4</sup> See [1], Section 2, for the corresponding result in the isothermal theory.

<sup>5</sup> We use the conventional notation  $A \times B$  for the cartesian product of a set  $A$  and a set  $B$ .

where  $R$  stands for the closure of  $\Omega$ , i.e. for the union of  $\Omega$  and its boundary. Finally, in the case of the standard mixed boundary-value problem, the boundary conditions become

$$\left. \begin{aligned} u_1 &= u_1^b \quad \text{on } B_1 \times (-\infty, \infty), \\ S_1 &= \sigma_{1j} n_j = S_1^b \quad \text{on } B_2 \times (-\infty, \infty). \end{aligned} \right\} (2.9)$$

Here  $B_1$  and  $B_2$  are complementary subsets of the boundary  $B$  of  $\Omega$ ,  $n_j$  is the outward unit normal of  $B$ , whereas  $u_1^b$  and  $S_1^b$  are prescribed surface displacements and surface tractions, respectively.

In the absence of an explicit statement to the contrary, the point sets  $R, B_1, B_2$  will henceforth be assumed to be independent of the time. Further, we shall suppose from here on that  $R$  is a bounded regular region of space<sup>6</sup> and that  $B_1, B_2$  are both integrable.

The problem under consideration thus consists in determining field histories  $u_1, \varepsilon_{1j}, \sigma_{1j}$  which — for given  $R, B_1, B_2$ , known  $G_1, G_2, \alpha$ , and prescribed  $F_1, \Theta, u_1^b, S_1^b$  — satisfy the field equations (2.1), (2.2), (2.3), (2.6) in  $\Omega \times (-\infty, \infty)$  and meet the initial conditions (2.8), as well as the boundary conditions (2.9). The temperature field history  $\Theta$  may, in particular, be specified as the solution of an independent heat-conduction problem.

For future economy we now adopt the following definition of a

<sup>6</sup>By a "regular region of space" we mean a region whose boundary consists of a finite number of non-intersecting "closed regular surfaces", the latter term being used in the sense of Kellogg [2].

Viscoelastic state. We say that the ordered array of field histories  $[u_1, \varepsilon_{1j}, \sigma_{1j}]$  belongs to the class of viscoelastic states on  $R \times (-\infty, \infty)$  corresponding to the data  $G_1, G_2, \alpha, F_1, \Theta$ , and write

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] \in \mathcal{V}[G_1, G_2, \alpha, F_1, \Theta] \text{ on } R \times (-\infty, \infty) \quad (2.10)$$

if:

- (a)  $G_p (\beta=1,2)$  vanishes on  $(-\infty, 0)$ , is twice continuously differentiable on  $[0, \infty)$ , and  $G_p(0) > 0$ ;
- (b)  $u_1, \varepsilon_{1j}, \sigma_{1j}, F_1, \Theta$  vanish on  $R \times (-\infty, 0)$  and are continuous on  $R \times [0, \infty)$ ,  $\Theta$  being once — and  $u_1$  three times continuously differentiable on  $R \times [0, \infty)$ ;
- (c) equations (2.1), (2.2), (2.3), (2.6) hold in  $R \times (-\infty, \infty)$ .

If, in particular,  $\Theta=0$  on  $R \times (-\infty, \infty)$ , we say that the state

$[u_1, \varepsilon_{1j}, \sigma_{1j}]$  is isothermal and write

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] \in \mathcal{V}_I[G_1, G_2, F_1] \text{ on } R \times (-\infty, \infty). \quad (2.11)$$

Requirements (a), (b), (c) are evidently partly redundant but mutually consistent. Also, the smoothness assumptions contained in (a), (b) could be relaxed (especially as far as the time-dependence is concerned) at the expense of more elaborate regularity hypotheses. Such refinements would, moreover, necessitate occasional supplementary smoothness assumptions in subsequent theorems and would merely tend to detract from the main purpose of the present paper.

It is essential to observe that the field histories involved in the preceding definition of a viscoelastic state may

evidently exhibit finite jump discontinuities at  $t=0$ . Although such singular field histories are physically unrealistic — particularly within the context of the quasi-static theory — they play an important part in the theory of integration of the governing field equations because of Duhamel's principle. One of the advantages derived from the formulation of the constitutive relations (2.6) in terms of Stieltjes convolutions lies in the fact that it permits a systematic treatment of the relevant discontinuities and enables one to avoid the usual purely formal manipulations with Dirac's delta function.

If (2.10) holds, the stress-strain law (2.6) admits the conventional Riemann integral representation (cf. [1], Theorem 3.4), valid for  $(\underline{x}, t)$  in  $R^N \times [0, \infty)$ ,

$$\left. \begin{aligned} s_{1j}(\underline{x}, t) &= G_1(t) \overset{\circ}{e}_{1j}(\underline{x}) + \int_0^t G_1(t-t') \dot{e}_{1j}(\underline{x}, t') dt', \\ \sigma_{kk}(\underline{x}, t) &= G_2(t) [\overset{\circ}{e}_{kk}(\underline{x}) - 3\alpha \overset{\circ}{\Theta}(\underline{x})] \\ &\quad + \int_0^t G_2(t-t') [\dot{e}_{kk}(\underline{x}, t') - 3\alpha \dot{\Theta}(\underline{x}, t')] dt'. \end{aligned} \right\} \quad (2.12)$$

Here, as in the sequel,  $\dot{f}$  stands for the first time derivative of a function  $f$  of position and time, whereas

$$\overset{\circ}{f}(\underline{x}) = f(\underline{x}, 0). \quad (2.13)$$

We consider next the case of an elastic solid. To this end let  $h$  henceforth denote the Heaviside unit step function defined by

$$\left. \begin{aligned} h(t) &= 0 \text{ for } -\infty < t < 0, \\ h(t) &= 1 \text{ for } 0 \leq t < \infty. \end{aligned} \right\} \quad (2.14)$$

It is immediate from (2.12) that for the particular choice

$$G_1 = 2\mu h, \quad G_2 = 3\kappa h, \quad (2.15)$$

where  $\mu$  and  $\kappa$  are constants, the stress-strain relations (2.6) pass over into

$$s_{1j} = 2\mu e_{1j}, \quad \sigma_{kk} = 3\kappa(\epsilon_{kk} - 3\alpha\Theta). \quad (2.16)$$

Equations (2.16), in view of (2.3), are equivalent to Hooke's law (modified to account for thermal expansion), provided  $\mu$  and  $\kappa$  are the shear modulus and the bulk modulus of the elastic material, respectively. This remark reveals the role of the classical quasi-static theory of thermoelasticity as a special case of the theory of thermo-viscoelasticity under consideration and motivates the following definition.

Quasi-static elastic state. If (2.10) holds and the relaxation functions obey (2.15), in which  $\mu$  and  $\kappa$  are (positive) constants, we say that  $[u_1, \epsilon_{1j}, \sigma_{1j}]$  belongs to the class of quasi-static elastic states on  $R \times (-\infty, \infty)$  corresponding to the data  $\mu, \kappa, \alpha, F_1, \Theta$ , and write

$$[u_1, \epsilon_{1j}, \sigma_{1j}] \in \mathcal{E}_Q[\mu, \kappa, \alpha, F_1, \Theta] \text{ on } R \times (-\infty, \infty). \quad (2.17)$$

It is clear from the above definition that a quasi-static elastic state which fails to vanish identically on  $R \times (-\infty, \infty)$ , cannot be independent of time on this entire space-time domain. For, (2.17) implies that  $u_1, \epsilon_{1j}, \sigma_{1j}$  must meet the

initial conditions (2.8). Since we shall need to deal also with purely position-dependent solutions of the fundamental field equations of thermoelasticity theory, we find it convenient to introduce the notion of a

Stationary elastic state. We say that  $[u_1, \varepsilon_{1j}, \sigma_{1j}]$  belongs to the class of stationary elastic states on  $R$  corresponding to the data  $\mu, \kappa, \alpha, F_1, \Theta$ , and write

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] \in \mathcal{E}[\mu, \kappa, \alpha, F_1, \Theta] \text{ on } R \quad (2.18)$$

if:

- (a)  $\mu$  and  $\kappa$  are (not necessarily real) constants;
- (b)  $u_1, \varepsilon_{1j}, \sigma_{1j}, F_1, \Theta$  are functions of position continuous on  $R$ ,  $\Theta$  being once — and  $u_1$  three times continuously differentiable on  $R$ ;
- (c) equations (2.1), (2.2), (2.3), (2.16) hold in  $\mathcal{R}$ .

The reason why we do not restrict  $\mu$  and  $\kappa$  to be real constants in the present instance will become apparent at the end of Section 4. Meanwhile we note that (2.17) implies, for each fixed  $t$  in  $(-\infty, \infty)$ ,

$$[u_1(\cdot, t), \varepsilon_{1j}(\cdot, t), \sigma_{1j}(\cdot, t)] \in \mathcal{E}[\mu, \kappa, \alpha, F_1(\cdot, t), \Theta(\cdot, t)] \text{ on } R \quad (2.19)^7$$

so that every quasi-static elastic state on  $R \times (-\infty, \infty)$  may be regarded as a one-parameter family (with time as the parameter) of stationary elastic states on  $R$  corresponding to the same

<sup>7</sup> If  $f$  is a function of position and time defined on  $R \times (-\infty, \infty)$ , we write  $f(\cdot, t)$  for the function of position defined on  $R$  that results from the mapping  $f$  by holding the time fixed in  $(-\infty, \infty)$ .

elastic constants and to the appropriate family of body-force and temperature distributions. This observation reflects the fact that the time in quasi-static elasticity theory plays the role of a mere parameter, whereas the quasi-static theory of viscoelasticity is genuinely time-dependent.

We have so far based the formulation of boundary-value problems in thermo-viscoelasticity theory upon the relaxation integral law (2.6). On the assumption that  $G_\beta(\beta=1,2)$ ,  $\epsilon_{ij}$ , and  $\sigma_{ij}$  meet the conditions set down under (a),(b) in the definition of a viscoelastic state, the stress-strain relations (2.6) may be inverted. This leads ([1], Theorem 3.3) to the equivalent creep integral law

$$e_{ij} = s_{ij} * dJ_1, \quad \epsilon_{kk} = \sigma_{kk} * dJ_2 + 3\alpha\Theta, \quad (2.20)$$

where  $J_1$  and  $J_2$  are the creep compliances in shear and isotropic compression, respectively. Moreover, the two pairs of response functions  $G_\beta$  and  $J_\beta$  ( $\beta=1,2$ ) are linked by the relation

$$G_\beta * dJ_\beta = h \quad \text{on } (-\infty, \infty) \quad (2.21)$$

or, using a notation adopted in [1] (Theorem 1.3) for the "Stieltjes inverse" of a function of time,

$$J_\beta = G_\beta^{-1} \quad \text{on } (-\infty, \infty). \quad (2.22)$$

Equation (2.21) implies further that

$$G_\beta J_\beta(t) + \int_0^t \dot{G}_\beta(t-t') J_\beta(t') dt' = 1 \quad \text{for } 0 \leq t < \infty. \quad (2.23)$$

The familiar physical significance of the relaxation moduli and of the creep compliances is immediate from (2.12) and

its creep counterpart. Thus

$$e_{1j} = h \text{ on } R \times (-\infty, \infty) \text{ implies } s_{1j} = G_1 \text{ on } R \times (-\infty, \infty), \quad (2.24)$$

$$s_{1j} = h \text{ on } R \times (-\infty, \infty) \text{ implies } e_{1j} = J_1 \text{ on } R \times (-\infty, \infty). \quad (2.25)$$

Analogous interpretations apply to  $G_2$  and  $J_2$ . If  $f$  is a function of position and time that possesses a Laplace transform with respect to time, we write

$$\bar{f}(\underline{x}, \eta) = \mathcal{L}\{f(\underline{x}, t); \eta\} = \int_0^{\infty} f(\underline{x}, t) \exp(-\eta t) dt, \quad (2.26)$$

in which  $\eta$  designates the transform parameter. Accordingly, supposing  $G_\beta$  and  $J_\beta$  ( $\beta=1,2$ ) to be of exponential order as  $t \rightarrow \infty$ , one draws from (2.23) in conjunction with the convolution theorem for the Laplace transform that

$$\bar{G}_\beta(\eta) \bar{J}_\beta(\eta) = \frac{1}{\eta^2} \quad (\beta=1,2). \quad (2.27)$$

We turn finally to stress-strain relations in differential equation form, i.e. to what is traditionally referred to as the differential operator law of linear viscoelasticity. For this purpose we first make the following notational agreements. If  $f$  is a (suitably smooth) function of position and time defined on  $R \times (-\infty, \infty)$ , we denote its  $n$ -th partial time derivative by

$$f^{(n)} \equiv D^n f \quad (n=0,1,2,\dots), \quad (2.28)$$

$D$  being the time-derivative operator. Further, we write  $f^{(n)0}$  for the function of position defined by

$$f^{(n)0}(\underline{x}) = f^{(n)}(\underline{x}, 0+) \text{ for } \underline{x} \text{ in } R. \quad (2.29)$$

The differential operator law now becomes

$$P_1(D)s_{1j} = Q_1(D)e_{1j}, \quad P_2(D)\sigma_{kk} = Q_2(D)[\varepsilon_{kk} - 3\alpha\Theta], \quad (2.30)$$

provided  $P_\beta(D), Q_\beta(D)$  ( $\beta=1,2$ ) are the linear differential operators

$$P_\beta(D) = \sum_{n=0}^{N_\beta} p_{\beta;n} D^n, \quad Q_\beta(D) = \sum_{n=0}^{N_\beta} q_{\beta;n} D^n \quad (\beta=1,2). \quad (2.31)$$

Here  $N_\beta$  ( $\beta=1,2$ ) is a non-negative integer and the coefficients  $p_{\beta;n}, q_{\beta;n}$  are constant response parameters for a given (temperature-independent) material. Moreover, we may evidently assume that either  $p_{n;\beta} \neq 0$  or  $q_{n;\beta} \neq 0$  when  $n = N_\beta$  ( $\beta=1,2$ ) so that, for fixed  $\beta$ , at least one of the operators in (2.31) has the degree  $N_\beta$ . The differential equations (2.30) are to be met in  $\mathbb{R}_x(0, \infty)$  and must be accompanied by the following initial conditions,

which are valid on  $R$ :

$$\left. \begin{aligned} \sum_{n=r}^{N_1} p_{1;n} s_{1j}^{(n-r)} &= \sum_{n=r}^{N_1} q_{1;n} e_{1j}^{(n-r)} \quad (r=1,2,\dots,N_1), \\ \sum_{n=r}^{N_2} p_{2;n} \sigma_{kk}^{(n-r)} &= \sum_{n=r}^{N_2} q_{2;n} [\varepsilon_{kk}^{(n-r)} - 3\alpha\Theta^{(n-r)}] \quad (r=1,2,\dots,N_2). \end{aligned} \right\} \quad (2.32)$$

As is clear from an elementary extension of Theorem 4.1 in [1], equations (2.30), (2.32) are implied by (2.8) and the relaxation integral law (2.6) or the creep integral law (2.20) provided  $\varepsilon_{1j}, \sigma_{1j}$ , and  $\Theta$  are sufficiently smooth and according as the relaxation moduli or the creep compliances exhibit the degeneracies characteristic of a finite spectrum of relaxation or retardation times. In the first case  $p_{n;\beta} \neq 0$  for  $n=N_\beta$  ( $\beta=1,2$ ),

whereas in the second instance  $q_{n;\beta} \neq 0$  for  $n = N_\beta$  ( $\beta=1,2$ ). The physical significance of the initial conditions (2.32) was established in [1](Theorem 4.2) for the isothermal case<sup>8</sup>. Thus suppose  $\Theta$  vanishes identically and hold  $\underline{x}$  fixed. Then, roughly speaking, (2.32) are necessary and sufficient that every pair of local histories  $\varepsilon_{1j}(\underline{x}, \cdot)$ ,  $\sigma_{1j}(\underline{x}, \cdot)$  which vanishes on  $(-\infty, 0)$ , satisfies (2.30) on  $(0, \infty)$ , and exhibits finite jump discontinuities at  $t=0$ , be the limit of a pair of history sequences which vanishes on  $(-\infty, 0)$ , obeys the same stress-strain relations (2.30), and has time derivatives of the orders entering (2.30) that are continuous on  $(-\infty, \infty)$ .

If an integral law (2.6) is reducible to a differential operator law (2.30), then the Laplace transforms of the relaxation functions exist, are necessarily rational, and are given by ([1], Theorem 4.8)

$$\bar{G}_\beta(\eta) = \frac{Q_\beta(\eta)}{\eta P_\beta(\eta)} \quad (\beta=1,2). \quad (2.33)$$

Analogously, the reducibility of a creep integral law (2.20) implies

$$\bar{J}_\beta(\eta) = \frac{P_\beta(\eta)}{\eta Q_\beta(\eta)} \quad (\beta=1,2). \quad (2.34)$$

Additional results concerning the transition from an integral to a differential operator law, and vice versa, may be found in [1] (Section 4).

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<sup>8</sup> See also Boley and Weiner [3], Art. 15.6, for a related discussion.

Since materials with a finite relaxation or retardation spectrum admit the familiar model representation in terms of easily visualized finite networks of springs and dashpots, the bulk of the existing literature on viscoelasticity has favored the differential operator law over stress-strain relations in integral form. It is well to bear in mind, however, that such a preference is not warranted on the grounds of theoretical generality; nor is it justified from a practical point of view since an adequate description of the behavior of actual viscoelastic solids over any appreciable range of time (or frequency) ordinarily necessitates the use of differential operators of a comparatively high order.

3. Temperature-independent mechanical response: general implications.

We turn next to general results concerning the fundamental field equations and boundary-value problems discussed in the preceding section. In this connection we shall limit our attention to materials governed by the relaxation integral law (2.6); analogous conclusions apply to the creep integral law (2.20) and to the differential operator law (2.30). To facilitate the task at hand we first establish a link between the theory of thermo-viscoelasticity under present consideration and the corresponding isothermal theory, which is supplied by the

Body-force analogy. Suppose

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] \in \mathcal{V}[G_1, G_2, \alpha, F_1, \Theta] \text{ on } R \times (-\infty, \infty) \quad (3.1)$$

and

$$u_1 = u_1^b \text{ on } B_1 \times (-\infty, \infty), \quad S_1 = S_1^b \text{ on } B_2 \times (-\infty, \infty). \quad (3.2)$$

Further, let  $\tilde{u}_1, \tilde{\varepsilon}_{1j}, \tilde{\sigma}_{1j}, \tilde{F}_1$  be field histories defined on  $R \times (-\infty, \infty)$  by means of

$$\left. \begin{aligned} \tilde{u}_1 &= u_1, \quad \tilde{\varepsilon}_{1j} = \varepsilon_{1j}, \\ \tilde{\sigma}_{1j} &= \sigma_{1j} + \alpha \delta_{1j} \Theta * dG_2, \quad \tilde{F}_1 = F_1 - \alpha \Theta_{,1} * dG_2. \end{aligned} \right\} \quad (3.3)$$

Then

$$[\tilde{u}_1, \tilde{\varepsilon}_{1j}, \tilde{\sigma}_{1j}] \in \mathcal{V}_I[G_1, G_2, \tilde{F}_1] \text{ on } R \times (-\infty, \infty) \quad (3.4)$$

and

$$\tilde{u}_1 = u_1^b \text{ on } B_1 \times (-\infty, \infty), \quad \tilde{S}_1 = S_1^b + \alpha n_1 \Theta * dG_2 \text{ on } B_2 \times (-\infty, \infty). \quad (3.5)$$

Conversely, if  $\Theta$  has the same properties as in the definition of a viscoelastic state, then (3.3), (3.4), (3.5) imply (3.1) and (3.2).

The truth of the foregoing theorem is readily inferred from the underlying definitions of viscoelastic and isothermal viscoelastic states with the aid of a known result ([1], Theorem 1.6) on the space differentiation of Stieltjes convolutions. The theorem yields as a special case the well-known body-force analogy of thermoelasticity<sup>9</sup> if the relaxation functions obey (2.15). Although the generalized analogy under discussion,

<sup>9</sup> See, for example, [3], Art. 3.3.

which reduces the mixed boundary-value problem of thermo-visco-elasticity to the standard mixed problem in the isothermal theory, is of no practical usefulness as far as the actual solution of specific problems is concerned, its theoretical interest is considerable. For, the analogy enables one to obtain economically extensions to non-isothermal conditions of theorems available in the isothermal theory. We now cite some of the more important conclusions reached in this manner.

Volterra's [4] theorem<sup>10</sup> concerning the uniqueness of the solution to the mixed isothermal problem, in conjunction with the body-force analogy, at once furnishes the

Uniqueness theorem. Suppose

$$\left. \begin{aligned} [u_1, \varepsilon_{1j}, \sigma_{1j}] &\in V[g_1, g_2, \alpha, F_1, \Theta] \text{ on } R \times (-\infty, \infty), \\ [u_1', \varepsilon_{1j}', \sigma_{1j}'] &\in V[g_1, g_2, \alpha, F_1, \Theta] \text{ on } R \times (-\infty, \infty), \end{aligned} \right\} (3.6)$$

and let

$$u_1 = u_1' \text{ on } B_1 \times (-\infty, \infty), \quad S_1 = S_1' \text{ on } B_2 \times (-\infty, \infty). \quad (3.7)$$

Then

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] = [u_1', \varepsilon_{1j}', \sigma_{1j}'] + [w_1, 0, 0] \text{ on } R \times (-\infty, \infty), \quad (3.8)$$

where  $w_1 = 0$  on  $R \times (-\infty, 0)$  and  $w_1$  represents an (infinitesimal) rigid motion of the entire body on  $R \times [0, \infty)$ .

According to (a) in the definition of a viscoelastic state, the relaxation functions entering (3.6) are required to

<sup>10</sup> See [1], Section 8, for a slightly more general version and a more detailed proof of Volterra's result in the special case of an isotropic solid.

satisfy the inequalities

$$\overset{\circ}{G}_\beta > 0 \quad (\beta=1,2), \quad (3.9)$$

i.e. must possess positive initial values. The striking fact that no restrictions (apart from a smoothness assumption) need be imposed on the subsequent behavior of  $\overset{\circ}{G}_\beta$  in order to insure uniqueness, will be made plausible shortly.

A characterization of the initial field distributions appropriate to a viscoelastic state is supplied by the following theorem, which is an elementary consequence of Theorem 6.1 in [1] and of the body-force analogy.

Initial response. Let  $[u_1, \varepsilon_{1j}, \sigma_{1j}]$  meet (2.10). Then

$$[\overset{\circ}{u}_1, \overset{\circ}{\varepsilon}_{1j}, \overset{\circ}{\sigma}_{1j}] \in \mathcal{A}[\mu, \kappa, \alpha, \overset{\circ}{F}_1, \overset{\circ}{\Theta}] \text{ on } R \quad (3.10)$$

with

$$\mu = \frac{1}{2} \overset{\circ}{G}_1, \quad \kappa = \frac{1}{3} \overset{\circ}{G}_2. \quad (3.11)$$

This result lends a precise meaning — within the context of thermo-viscoelasticity theory — to the familiar assertion that the initial response of a viscoelastic solid is elastic. Moreover, the preceding theorem enables one to determine directly the initial displacements, stresses, and strains belonging to the solution of a mixed boundary-value problem in thermo-viscoelasticity theory by solving a steady-state thermoelastic problem; the latter is governed by the initial body-force, temperature, and surface data of the original problem, as well as by the elastic constants (3.11). Similarly, Theorem 6.2 in [1] furnishes an

analogous characterization of the initial (right-hand) time derivatives of all existing orders belonging to the desired viscoelastic state. These initial time rates  $u_1^{(n)}, \varepsilon_{1j}^{(n)}, \sigma_{1j}^{(n)}$  ( $n=1,2,\dots$ ) may be found directly from the corresponding initial data by solving successively a sequence of steady-state thermoelastic problems, each of which is governed once again by the elastic constants defined in (3.11). Since, according to the uniqueness theorem of the classical (uncoupled) theory of thermoelasticity, the inequalities  $\mu > 0$ ,  $\kappa > 0$  are sufficient to guarantee the uniqueness of a suitably regular solution to the mixed problem, Volterra's uniqueness criterion (3.9) is not so surprising.

An additional conclusion regarding the time-dependence of viscoelastic states, that permits one to infer the smoothness of the state with respect to time from the corresponding smoothness of the field and surface data, is deducible from Theorem 6.4 in [1]. We proceed now to a theorem on the Position-dependence of viscoelastic states. Let  $[u_1, \varepsilon_{1j}, \sigma_{1j}]$  meet (2.10) and assume that

$$\nabla \cdot \underline{F} = 0, \nabla \wedge \underline{F} = 0, \nabla^2 \theta = 0 \text{ on } \mathbb{R} \times (-\infty, \infty). \quad (3.12)^{11}$$

Then throughout  $\mathbb{R} \times (-\infty, \infty)$ ,

$$\nabla^2(\nabla \cdot \underline{u}) = 0, \nabla^2(\nabla \wedge \underline{u}) = 0, \quad (3.13)$$

$$\nabla^4 u_1 = 0, \nabla^4 \varepsilon_{1j} = 0, \nabla^4 \sigma_{1j} = 0. \quad (3.14)$$

<sup>11</sup> Here, as in the sequel,  $\nabla$  is the usual spatial gradient operator, whereas  $\nabla \cdot$ ,  $\nabla \wedge$ , and  $\nabla^2$  designate the divergence, curl, and Laplacian operators, respectively.

Thus in the present circumstances, as in thermoelasticity theory, (3.12) imply that the dilatation and the rotation are harmonic, while the cartesian components of displacement, strain, and stress are biharmonic functions of position. It follows that all of these field histories possess continuous space derivatives of all orders. The above theorem is a consequence of Theorem 6.6 in [1].

The counterpart of Betti's elastostatic reciprocal relations in the isothermal theory of viscoelasticity given in [1] (Theorem 7.4), in view of the body-force analogy, the divergence theorem, and by virtue of Theorems 1.2, 1.6 in [1], leads to the following

Reciprocal theorem. Suppose

$$\left. \begin{aligned} [u_1, \varepsilon_{1j}, \sigma_{1j}] &\in \mathcal{V}[G_1, G_2, \alpha, F_1, \Theta] \text{ on } R \times (-\infty, \infty), \\ [u'_1, \varepsilon'_{1j}, \sigma'_{1j}] &\in \mathcal{V}[G_1, G_2, \alpha, F'_1, \Theta'] \text{ on } R \times (-\infty, \infty). \end{aligned} \right\} \quad (3.15)$$

Then, on  $(-\infty, \infty)$ ,

$$\begin{aligned} \int_B S_1 * du'_1 dA + \int_R F_1 * du'_1 dV + \alpha \int_R \Theta * d\varepsilon'_{11} * dG_2 dV = \\ \int_B S'_1 * du_1 dA + \int_R F'_1 * du_1 dV + \alpha \int_R \Theta' * d\varepsilon_{11} * dG_2 dV = \\ \int_R \sigma_{1j} * d\varepsilon'_{1j} dV + \alpha \int_R \Theta * d\varepsilon'_{11} * dG_2 dV = \\ \int_R \sigma'_{1j} * d\varepsilon_{1j} dV + \alpha \int_R \Theta' * d\varepsilon_{11} * dG_2 dV. \end{aligned} \quad (3.16)$$

It should be emphasized that the reciprocal relations obtained from the present theorem by specializing  $G_\beta$  ( $\beta=1,2$ ) in accordance with (2.15) and by setting  $\Theta=0$  on  $R \times (-\infty, \infty)$ , are distinct from the quasi-static adaptation of Betti's reciprocal relations in isothermal elastostatics. This distinction disappears if the body force and temperature field histories, as well as the surface data, of the two states in (3.15) are separable functions of position and time that share a common time-dependence. In this particular instance the convolutions entering (3.16) (e.g.  $S_1 * du_1'$ ) can be shown<sup>12</sup> to be reducible to ordinary products (e.g.  $S_1 u_1'$ ), and the resulting reciprocal relations are identical with those arising from the extension of Betti's theorem to quasi-static thermoelasticity theory.

The reciprocal theorem stated above may be used to derive formulas for the average strains induced in a (temperature-independent) viscoelastic solid by given body-force and temperature field histories and by specified surface tractions.<sup>13</sup> We include here merely a formula for the total volume change, which can be deduced directly from Theorem 7.6 in [1] with the aid of the body-force analogy and the divergence theorem.

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<sup>12</sup> Cf. Theorem 7.5 in [1].

<sup>13</sup> See [5] for a derivation of analogous results in the isothermal theory of anisotropic viscoelastic solids.

Volume change. From (2.10) follows, for  $t$  in  $(-\infty, \infty)$ ,

$$\Delta V(t) = [\Lambda * dJ_2](t) + 3\alpha \int_R \Theta(\underline{x}, t) dV, \quad (3.17)$$

provided  $J_2$  is the creep compliance corresponding to the relaxation modulus  $G_2$  and where

$$\Delta V(t) = \int_R \epsilon_{11}(\underline{x}, t) dV, \quad (3.18)$$

$$\Lambda(t) = \int_B x_1 S_1(\underline{x}, t) dA + \int_R x_1 F_1(\underline{x}, t) dV. \quad (3.19)$$

It is essential to observe that (3.17) may also be derived<sup>14</sup> solely from (3.18), (2.2), and the second of (2.6) by recourse to (2.21) and Theorems 1.2, 1.6 in [1]. Consequently, formula (3.17) is valid for any solid that is in a state of infinitesimal deformations and has a temperature-independent linearly viscoelastic dilatational response, regardless of the nature of its response in shear. If, in particular,  $S_1=0$  on  $B \times (-\infty, \infty)$  and  $F_1=0$  on  $R \times (-\infty, \infty)$ , (3.17) implies

$$\Delta V(t) = 3\alpha \int_R \Theta(\underline{x}, t) dV \text{ for } t \text{ in } (-\infty, \infty). \quad (3.20)$$

The same conclusion was also reached by Nowacki [7](Chapter XI) on the basis of a less direct argument. Thus, in the absence of surface tractions and body forces, the total volume change is the same as that which would arise from uninhibited thermal expansion.

<sup>14</sup> See [6], where such an alternative derivation is given for the special case of a material with a purely elastic dilatational response.

This well-known result of classical thermoelasticity theory<sup>15</sup> is accordingly seen to survive without modification under the present more general hypotheses.

The next theorem, which also pertains to vanishing surface tractions and body forces, follows from the appropriate specialization of a result established in [6] for a class of temperature-dependent viscoelastic materials.

Stress-free temperature field histories. Assume (2.10) holds and suppose  $S_1=0$  on  $B \times [0, \infty)$ ,  $F_1=0$  on  $R \times [0, \infty)$ . Then  $\sigma_{1j}=0$  on  $R \times (-\infty, \infty)$  if and only if

$$\Theta(\underline{x}, t) = a_0(t) + a_1(t)x_1 \text{ for } (\underline{x}, t) \text{ in } R \times [0, \infty), \quad (3.21)$$

where  $a_0, a_1$  are functions continuous on  $[0, \infty)$ .

An indication of the proof of the necessity of (3.21) if the body is to be free from stress, appeared prior to [6] in a note by Hilton [9]. Analogous two-dimensional theorems on temperature distributions that fail to induce thermal stresses (applicable to conditions of plane strain and generalized plane stress) are also to be found in [6]. For the familiar corresponding three-dimensional and two-dimensional results in thermoelasticity theory, which are implied by the theorems under present consideration, reference may be made to [3], Arts. 3.9, 4.9.

Before concluding this section we merely mention several additional implications of the body-force analogy. Thus, one may

<sup>15</sup> See, for example, [3], Art. 9.15. The result was reached by Hieke [8], who based his proof on an extension to thermoelasticity theory of Betti's reciprocal relations.

extend Duhamel's principle of the isothermal theory (see [1], Section 5) to the analysis of thermal stresses and deformations. Similarly, one may obtain analogous extensions of the variational principles of isothermal viscoelasticity theory.<sup>16</sup> Finally, the integral representations (in terms of Green's functions) of the solution to isothermal boundary-value problems established in [11] are readily generalized to include temperature effects within the current theoretical framework.

Some further consequences of the body-force analogy will be encountered in the succeeding section, which deals with results that have a direct bearing on the solution of boundary-value problems of the type formulated in Section 2.

4. Temperature-independent mechanical response: methods of integration, available solutions.

With a view toward alternative formulations of the boundary-value problems stated in Section 2, we now cite the appropriate displacement equations of equilibrium and stress equations of compatibility, which could be arrived at by a respective elimination of either the stresses and strains, or of the displacements and strains, among the governing fundamental system of field equations. Actually, these desired generalizations of the Cauchy equations of equilibrium and of the Beltrami-Michell equations of compatibility are immediate from their isothermal counterparts in Theorem 5.5 and Theorem 5.7 of [1] because of the body-force analogy.

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<sup>16</sup> See Gurtin [10] for a generalization to isothermal viscoelasticity theory of the classical, and of more recent, elastostatic variational principles.

Displacement equations of equilibrium. From (2.10) follows, on  $\mathbb{R} \times (-\infty, \infty)$ ,

$$u_{1,jj} * dG_1 + \frac{1}{3} u_{j,j1} * d(G_1 + 2G_2) + 2F_1 = 2\alpha_{\Theta,1} * dG_2 \quad (4.1)$$

or, equivalently,

$$\nabla^2 \underline{u} * dG_1 + \frac{1}{3} \nabla \nabla \cdot \underline{u} * d(G_1 + 2G_2) + 2\underline{F} = 2\alpha \nabla \Theta * dG_2. \quad (4.2)$$

Stress equations of compatibility. From (2.10) follows, on  $\mathbb{R} \times (-\infty, \infty)$ ,

$$\nabla^2 \sigma_{1j} * dJ_1 + \frac{1}{3} \sigma_{kk,1j} * d(2J_1 + J_2) = \Phi_{1j}, \quad (4.3)$$

where  $J_\beta$  is the creep compliance corresponding to  $G_\beta$  ( $\beta=1,2$ ),

$$\begin{aligned} \Phi_{1j} = & \delta_{1j} F_{k,k} * d\Omega - (F_{1,j} + F_{j,1}) * dJ_1 \\ & - \alpha \delta_{1j} \nabla^2 \Theta * dG_2 * d(\Omega + J_1) - \alpha_{\Theta,1j}, \end{aligned} \quad (4.4)$$

and

$$\Omega = J_1 * d(J_2 - J_1) * d(J_1 + 2J_2)^{-1}. \quad (4.5)^{17}$$

The mixed problem, when cast in terms of displacements alone, consists in finding a solution of (4.1) on  $\mathbb{R} \times (-\infty, \infty)$ , such that  $u_1 = 0$  on  $\mathbb{R} \times (-\infty, 0)$  and subject to the boundary conditions

$$u_1 = u_1^b \text{ on } B_1 \times (-\infty, \infty),$$

$$\begin{aligned} \left[ \frac{1}{2} (u_{1,j} + u_{j,1}) * dG_1 + \frac{1}{3} \delta_{1j} u_{k,k} * d(G_2 - G_1) - \alpha \delta_{1j} \Theta * dG_2 \right] n_j = S_1^b \quad (4.6) \\ \text{on } B_2 \times (-\infty, \infty), \end{aligned}$$

<sup>17</sup> Here, as on subsequent occasions, we use the notation for the Stieltjes inverse introduced in [1] (Theorem 1.3). Cf. (2.21), (2.22).

which follow from (2.9) by virtue of (2.1), (2.3), (2.6). On the other hand, if  $R$  is simply connected and  $B_2=B$  (surface tractions prescribed over the entire boundary) the unknown stresses are completely characterized by (2.2), (4.3), and the second of the boundary conditions (2.9).

Our next objective is to exhibit a particular solution of (4.2), corresponding to vanishing body forces and to a given temperature field history, by generalizing the familiar thermo-elastic potential, which appears to have been discovered originally by Borchardt [12].

Thermo-viscoelastic potential. Let  $G_\beta$  ( $\beta=1,2$ ) and  $\Theta$  meet conditions (a) and (b) in the definition of a viscoelastic state. Suppose  $\Theta$  is a (sufficiently smooth) real-valued function defined on  $\mathbb{R} \times (-\infty, \infty)$  that vanishes on  $\mathbb{R} \times (-\infty, 0)$  and satisfies

$$\nabla^2 \Theta = 3\alpha \Theta * dG_2 * d(2G_1 + G_2)^{-1} \quad (4.7)$$

in  $\mathbb{R} \times (-\infty, \infty)$ . Then throughout that domain the function  $u$  defined by

$$u = \nabla \Theta \quad (4.8)$$

satisfies (4.2), provided  $F=0$  on  $\mathbb{R} \times (-\infty, \infty)$ .

To confirm this assertion one need merely substitute from (4.8) into (4.2), use (4.7), and invoke the algebra and calculus of Stieltjes convolutions.<sup>18</sup> A suitably regular solution of the Poisson equation (4.7) may, in turn, be generated with the

<sup>18</sup> For related, though less explicit, particular solutions of (4.2) see Parkus [13](Chapter VI) and Nowacki [7](Chapter XI).

aid of a Newtonian potential, as is apparent from Lemma 9.1 in [1]. Indeed, if  $\rho$  designates the right-hand member of (4.7), such a solution is given by

$$\theta(\underline{x}, t) = - \frac{1}{4\pi} \int_R \frac{\rho(\underline{\xi}, t)}{|\underline{x} - \underline{\xi}|} dV_{\underline{\xi}} \text{ for all } (\underline{x}, t) \text{ in } R \times (-\infty, \infty). \quad (4.9)$$

The last theorem evidently supplies yet another means of reducing the boundary-value problem of Section 2 to a mixed problem in the isothermal theory. Further, the present theorem, in conjunction with Theorem 9.2 in [1], leads to the following generalization to thermo-viscoelasticity theory of the Papkovitch-Neuber stress functions in classical elastostatics.

Generalized Papkovitch-Neuber solution. Let  $G_{\beta}$  ( $\beta=1,2$ ),  $\Theta, \theta$  meet the hypotheses of the preceding theorem and let  $F$  conform to (b) in the definition of a viscoelastic state. Suppose  $\varphi$  and  $\psi$  are (sufficiently smooth) functions defined on  $R \times (-\infty, \infty)$ , both of which vanish on  $R \times (-\infty, 0)$  and satisfy

$$\nabla^2 \varphi = - \frac{1}{2} \underline{x} \cdot \underline{H}, \quad \nabla^2 \psi = \frac{1}{2} \underline{H}, \quad (4.10)$$

with

$$\underline{H} = \underline{F} * dG_1^{-1} * d(2G_1 + G_2)^{-1}, \quad (4.11)$$

in  $R \times (-\infty, \infty)$ . Then throughout that domain the function  $\underline{u}$  defined by

$$\underline{u} = \nabla \theta + \nabla(\varphi + \underline{x} \cdot \psi) * d(G_1 + 2G_2) - 4\psi * d(2G_1 + G_2) \quad (4.12)$$

satisfies (4.2).

In view of Theorem 9.4 in [1], the foregoing solution of the displacement equations of equilibrium is complete in the sense that every suitably regular solution of (4.2) admits the representation (4.7), (4.10), (4.12). Consequently, the introduction of the displacement potentials (stress functions)  $\Theta, \varphi$ , and  $\underline{\psi}$  reduces the mixed boundary-value problem to the determination of appropriate solutions of Poisson's equation. On the other hand, since  $\varphi$  and  $\underline{\psi}$  enter (4.12) under Stieltjes convolutions and because of the structure of the second of (4.6), the application of the boundary conditions to the generating stress functions will ordinarily lead to a system of simultaneous integral equations.

We turn now to the well-known correspondence principle that links the linear theories of viscoelasticity and elasticity. This principle, which is of paramount importance to the solution of the class of boundary-value problems with which we are concerned, may — with the aid of the state definitions adopted in Section 2 — be phrased in the following concise form.

Correspondence principle. Suppose

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] \in \mathcal{V}[G_1, G_2, \alpha, F_1, \Theta] \text{ on } R \times (-\infty, \infty) \quad (4.13)$$

and

$$u_1 = u_1^b \text{ on } B_1 \times (-\infty, \infty), \quad S_1 = S_1^b \text{ on } B_2 \times (-\infty, \infty). \quad (4.14)$$

Assume that  $G_\beta(t)$  ( $\beta=1,2$ ),  $u_1(\underline{x},t)$ ,  $\varepsilon_{1j}(\underline{x},t)$ ,  $F_1(\underline{x},t)$ , and  $\Theta(\underline{x},t)$ , for every  $\underline{x}$  in  $R$ , are all of the exponential order  $O(\exp(s_0 t))$  as  $t \rightarrow \infty$ , where  $s_0$  is a (real) constant. Then, for each  $\eta$  with  $\text{Re}(\eta) > s_0$ ,

$$[\bar{u}_1(\cdot, \eta), \bar{\varepsilon}_{1,j}(\cdot, \eta), \bar{\sigma}_{1,j}(\cdot, \eta)] \in \mathcal{C}[\mu(\eta), \kappa(\eta), \alpha, \bar{F}_1(\cdot, \eta), \bar{\Theta}(\cdot, \eta)] \text{ on } R, \quad (4.15)^{19}$$

where

$$\mu(\eta) = \frac{1}{2} \eta \bar{a}_1(\eta), \quad \kappa(\eta) = \frac{1}{3} \eta \bar{a}_2(\eta), \quad (4.16)$$

and further

$$\bar{u}_1(\cdot, \eta) = \bar{u}_1^b(\cdot, \eta) \text{ on } B_1, \quad \bar{\varepsilon}_1(\cdot, \eta) = \bar{\varepsilon}_1^b(\cdot, \eta) \text{ on } B_2. \quad (4.17)$$

The theorem just stated evidently implies that if the solution to the thermo-viscoelastic problem governed by (4.13), (4.14) is Laplace-transformable, it must coincide with the inverse transform of the solution to the one-parameter family of stationary thermoelastic boundary-value problems characterized by (4.15), (4.16), (4.17). Accordingly the correspondence principle yields a reduction of the original problem to one in steady-state thermoelasticity theory.

To confirm the principle one need merely remove the time-dependence from the field equations and boundary conditions appropriate to a viscoelastic state by applying the Laplace transform to (2.1), (2.2), (2.3), (2.6), (2.9) and compare the resulting system of equations with (2.1), (2.2), (2.3), (2.16), and (2.9). In particular, since the stress-strain relations (2.6), because of (2.8), imply (2.12), the convolution theorem for Laplace transforms furnishes

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<sup>19</sup> Recall Footnote No. 7 and the notation for the Laplace transform introduced in (2.26).

$$\left. \begin{aligned} \bar{s}_{1j}(\cdot, \eta) &= \eta \bar{g}_1(\eta) \bar{e}_{1j}(\cdot, \eta), \\ \bar{\sigma}_{kk}(\cdot, \eta) &= \eta \bar{g}_2(\eta) [\bar{e}_{kk}(\cdot, \eta) - 3\alpha \bar{\Theta}(\cdot, \eta)], \end{aligned} \right\} (4.18)$$

which, by virtue of (2.16), account for the "elastic constants" (4.16).

Suppose now that the formulation of the original (thermo-viscoelastic) boundary-value problem is based on the differential operator law (2.30) together with the initial conditions (2.32), rather than on the relaxation integral law (2.6). In this case the conclusions (4.15), (4.17) continue to hold true (in the presence of adequate regularity assumptions on the field histories involved), provided (4.16) are replaced by

$$\mu(\eta) = \frac{1}{2} \frac{Q_1(\eta)}{P_1(\eta)}, \quad \kappa(\eta) = \frac{1}{3} \frac{Q_2(\eta)}{P_2(\eta)}, \quad (4.19)$$

as is suggested by (2.33). A direct proof of the correspondence principle applicable to differential operator laws may be conducted by means of an argument that is strictly analogous to the one we have outlined for the relaxation integral law. Note, however, that operating on (2.30) with the Laplace transform one arrives at

$$\left. \begin{aligned} \bar{s}_{1j}(\cdot, \eta) &= \frac{Q_1(\eta)}{P_1(\eta)} \bar{e}_{1j}(\cdot, \eta), \\ \bar{\sigma}_{kk}(\cdot, \eta) &= \frac{Q_2(\eta)}{P_2(\eta)} [\bar{e}_{kk}(\cdot, \eta) - 3\alpha \bar{\Theta}(\cdot, \eta)] \end{aligned} \right\} (4.20)$$

on  $R$ , for all  $\eta$  with a sufficiently large real part, only if the initial conditions (2.32) are met.<sup>20</sup> Thus (2.32), the physical

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<sup>20</sup> For details see the proof of Theorem 4.7 in [1].

significance of which was discussed in Section 2, are precisely the conditions needed to justify the usual formal application of the Laplace transform to differential operator laws, in which jump discontinuities in  $\sigma_{ij}, \epsilon_{ij}$  (and in their relevant time derivatives) at  $t=0$  are either tacitly ruled out or else (2.32) are tacitly assumed to hold. This issue was apparently first recognized by Corneliussen and Lee [14] and was further clarified by Boley and Weiner [3] (Article 15.6); it is treated rigorously in considerable detail in [1].

The preceding analogy between boundary-value problems in the linear theories of viscoelasticity and elasticity has its origins in a paper by Alfrey [15]. Alfrey's form of the correspondence principle, which is based on the differential operator law, is confined to incompressible solids and to isothermal conditions. A limited extension of Alfrey's analogy to compressible solids is due to Tsien [16], who abandoned the assumption of incompressibility and postulated instead an artificial (physically unrealistic) connection between the deviatoric and the dilatational response of the material. Integral transforms were first brought to bear on the question raised in [15] by Read [17], who reduced the viscoelastic problem for a compressible solid to an elastostatic problem with the aid of the Fourier transform. Although the stress-strain law employed in [17] involves only three independent differential operators, this superfluous restriction does not affect the generality of Read's argument. Brull [18] arrived at an analogous correspondence

principle for a medium with a continuous spectrum of relaxation times by means of the Laplace transform and on the basis of the relaxation integral law; in [17] the volumetric response is (unnecessarily) assumed to be purely elastic. Lee [19] deduced the counterpart of Brull's version of the correspondence principle for the general differential operator law. Hilton, Hassan, and Russell [20] allowed for thermal expansion within the framework of Alfrey's analogy,<sup>21</sup> whereas Lee's scheme was similarly extended to thermo-viscoelasticity theory in [21]. Additional related results will be referred to later on. We note, in passing, that the initial conditions (2.32), which must supplement the differential operator law (2.30), are not mentioned in [15],[16],[17],[19], and [20], while the particular initial conditions presupposed in [21] are unduly stringent and hence misleading.

The extended correspondence principle was applied in [21] to the solution of a particular spatial boundary-value problem and has since been further exploited in the quasi-static analysis of thermal stresses and deformations for linear viscoelastic solids with temperature-independent properties. Specific examples and pertinent references may be found in [3],[7], and [13]. Nowacki's [7] monograph supplies a convenient guide to recent Polish papers on problems of the type formulated in Section 2. Among these we cite investigations by Nowacki [22],[23] and by Sokolowski [24]. A study by Shaffer and Levitsky [25] should also be referred to in the present context. Additional

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<sup>21</sup> The inclusion of the thermal expansion term is, strictly speaking, inconsistent with the accompanying assumption of incompressibility.

studies, which aim primarily at dynamic thermal effects in temperature-independent linear viscoelastic solids, will be mentioned in Section 7.

5. Temperature-dependent response: formulation of boundary-value problems for thermo-rheologically simple materials.

It is evident from the preceding section, that the quasi-static analysis of thermal stresses and deformations within the linear theory of homogeneous, isotropic viscoelastic solids, and in the absence of thermo-mechanical coupling, presents no essential difficulties, provided the mechanical response of the material is assumed to be independent of the temperature. Unfortunately, as was pointed out in the Introduction, such a treatment of the problem is remote from physical reality unless the temperature range involved is exceedingly small. The remainder of this paper is chiefly concerned with the modifications arising in the theory discussed so far if the relaxation moduli in (2.6), the creep compliances in (2.20), and the material response parameters in (2.30), (2.31) are themselves temperature-dependent.

The importance of effects that stem from the temperature-dependence of the viscoelastic properties was emphasized and illustrated relatively early by Freudenthal [26], [27], [28], [29]. Special problems concerning temperature-dependent linear viscoelastic solids were also considered by Hilton, Hassan, and Russell [20], as well as by Hilton [30], Rongred [31], Weiner and Mechanic [32], Landau, Weiner, and Zwicky [33], and Aggarwala [34]. All of the

problems treated in the publications just listed concern either infinite plates or (solid or hollow) spheres and circular cylinders, the field data and boundary conditions being such as to preclude the presence of more than a single space coordinate. Further, the underlying stress-strain relations correspond to degenerate cases of the differential operator law. In fact, the solids considered exhibit Maxwell, Kelvin, or Standard Linear behavior in shear, while the volumetric response is ordinarily taken to be either elastic or incompressible. Finally, the temperature-dependence of the material is in all cases, with the exception of [34], confined to the viscosity parameters and its specific choice is usually in part a matter of analytical expediency. Of particular interest is Rongved's [31] explicit solution (in closed elementary form) for the transient thermal stresses in an elastically compressible Maxwell sphere since it accommodates an arbitrary radially symmetric temperature field history and presupposes no restrictions upon the temperature-dependence of the shear viscosity.

A systematic scheme for including the influence of temperature on the response characteristics of a viscoelastic solid in the analysis of thermal stresses was initiated by Morland and Lee [35], who took as their point of departure the temperature-time equivalence hypothesis originally proposed by Leaderman [36] and subsequently introduced in a slightly different form by Ferri [37]. According to this postulate the

mechanical response of the material is affected by a uniform temperature change only within a uniform change of the time scale; the response is uniformly accelerated or slowed down depending on whether the temperature is increased or lowered. Materials obeying the temperature-time equivalence postulate are said to be "thermo-rheologically simple" in the terminology of Schwarzl and Staverman [38] who, in their expository article [39], cite experimental evidence in support of the postulate, which appears to be in remarkably good agreement with tests performed on a variety of high polymers over considerable temperature ranges. The analytical framework developed in [35] was further explored in [40].

We proceed now to a brief resumé of the theory of thermo-rheologically simple viscoelastic solids and in this connection consider first the required generalization of the relaxation integral law (2.6). Thus, let  $G_\beta(t)$  ( $\beta=1,2$ ) henceforth denote the values of the relaxation moduli at time  $t$  measured at the base temperature  $T_0$  and denote by  $G_\beta(t,T)$  the corresponding values measured at the (fixed) temperature  $T$ . Consequently

$$G_\beta(t, T_0) = G_\beta(t) \quad (\beta=1,2). \quad (5.1)$$

The temperature-time equivalence hypothesis may then be expressed analytically by

$$G_\beta(t, T) = G_\beta(\xi), \quad \xi = t\phi(T) \quad \text{for } (t, T) \text{ in } (-\infty, \infty) \times [T_1, T_2], \quad (5.2)$$

where  $[T_1, T_2]$  is the temperature range for which the postulate of

thermo-rheologically simple behavior is presumed to hold. Here  $\xi$  is the "reduced time", while  $\varphi$  represents the characteristic "shift function" of the material, which governs the contraction (or the stretching) of the time scale that results from the temperature change  $T-T_0$ . Clearly,

$$\varphi(T_0) = 1, \quad \varphi(T) > 0 \quad (T_1 \leq T \leq T_2) \quad (5.3)$$

and  $\varphi$  is evidently a steadily increasing function.

The constitutive law (2.6) must at present be confined in its validity to a material that is permanently maintained at the uniform temperature  $T_0$ . In these circumstances one gathers from (2.6) on setting  $\Theta=0$ , recalling (2.7), and making use of the commutativity of Stieltjes convolutions, that

$$\left. \begin{aligned} s_{1j}(\underline{x}, t) &= \int_{t'=-\infty}^t G_1(t-t') d\varepsilon_{1j}(\underline{x}, t'), \\ \sigma_{kk}(\underline{x}, t) &= \int_{t'=-\infty}^t G_2(t-t') d\varepsilon_{kk}(\underline{x}, t'). \end{aligned} \right\} \quad (5.4)$$

If the material is instead permanently at the uniform temperature  $T$ , the values  $G_p(t-t')$  in (5.4) — according to (5.2) — are to be replaced by  $G_p(\xi-\xi')$ , where  $\xi'=t'\varphi(T)$ . If, finally, the material is under the influence of a variable (time and position-dependent) temperature distribution  $T$  with values in  $[T_1, T_2]$ , (5.4) are in need of a two-fold additional amendment: first, the definition of the reduced time  $\xi$  must now be generalized to account for the cumulative effect of successive temperature changes;

second, one must allow once again for the influence of thermal expansion. This leads to the modified relaxation integral law

$$\begin{aligned} s_{ij}(\underline{x}, t) &= \int_{t'=-\infty}^t G_1(\xi - \xi') de_{ij}(\underline{x}, t'), \\ \sigma_{kk}(\underline{x}, t) &= \int_{t'=-\infty}^t G_2(\xi - \xi') d\epsilon_{kk}(\underline{x}, t') - 3\alpha \int_{t'=-\infty}^t G_2(\xi - \xi') d\Theta(\underline{x}, t'), \end{aligned} \quad (5.5)$$

with

$$\xi = \rho(\underline{x}, t) = \int_0^t \varphi(T(\underline{x}, t')) dt', \quad \xi' = \rho(\underline{x}, t'). \quad (5.6)^{22}$$

We note that the temperature enters (5.5) both through  $\Theta$  and through  $\xi, \xi'$ . In contrast to (2.6), the stress-strain relations (5.5) imply a nonlinear dependence of the local instantaneous stresses upon the local temperature history. It is apparent from (5.6), (5.3) that  $\rho(\underline{x}, \cdot)$  is a monotone increasing function of the time on  $(-\infty, \infty)$ , the inverse of which we denote by  $\omega(\underline{x}, \cdot)$ , whence

$$t = \omega(\underline{x}, \xi). \quad (5.7)$$

If  $f$  is a function of position and time, we shall consistently write  $\hat{f}$  for the function defined by

$$\hat{f}(\underline{x}, \xi) = f(\underline{x}, \omega(\underline{x}, \xi)). \quad (5.8)$$

<sup>22</sup>

Although this generalization of the reduced time, introduced in [35], is convincing on physical grounds, a rigorous deduction of (5.5) from (5.4) and the temperature-time equivalence postulate appears to be lacking.

By subjecting the variable of integration in (5.5) to the transformation  $t'=\omega(\underline{x},\xi')$  one may eliminate from (5.5) any explicit dependence upon the physical time in favor of the reduced time. In this manner, using the notations adopted in (2.7), (5.8) and involving once more the commutativity of Stieltjes convolutions, one arrives at the following particularly convenient version of the modified relaxation integral law:

$$\left. \begin{aligned} \hat{s}_{1j} &= \hat{e}_{1j} * dG_1 \\ \hat{\sigma}_{kk} &= (\hat{e}_{kk} - 3\alpha\hat{\Theta}) * dG_2 \end{aligned} \right\} (5.9)$$

As is apparent from (5.8), the convolutions in (5.9) are taken with respect to the reduced time rather than the physical time. We note that (5.9) have otherwise the same structure as (2.6).

Strictly analogous considerations apply to the generalization of (2.20) and ultimately lead to the modified creep integral law in the form

$$\left. \begin{aligned} \hat{e}_{1j} &= \hat{s}_{1j} * dJ_1, \\ \hat{e}_{kk} &= \hat{\sigma}_{kk} * dJ_2 + 3\alpha\hat{\Theta}, \end{aligned} \right\} (5.10)$$

which may also be deduced directly from (5.9). Finally, the same process of specialization that leads from the integral laws (2.6) or (2.20) to (2.30) now furnishes the modified differential operator law

$$P_1(\hat{D})\hat{s}_{1j} = Q_1(\hat{D})\hat{e}_{1j}, \quad P_2(\hat{D})\hat{\sigma}_{kk} = Q_2(\hat{D})[\hat{e}_{kk} - 3\alpha\hat{\Theta}], \quad (5.11)$$

where  $\hat{D}$  is the reduced-time derivative operator, i.e.

$$\hat{D} = \frac{\partial}{\partial \xi}, \quad (5.12)$$

while the polynomial operators  $P_\beta, Q_\beta$  ( $\beta=1,2$ ) retain their previous meaning. To (5.11) one must adjoin the initial conditions (2.32), which remain unaltered.

On referring (5.11) to the physical time  $t$  by means of (5.6), (5.7), (5.8), one is evidently led to a pair of differential equations which has the same structure as (2.30), except that the previously constant response parameters  $p_{\beta;n}, q_{\beta;n}$  are now functions of the temperature. It is also clear that these functions cannot be prescribed independently for a thermo-rheologically simple viscoelastic solid since the entire temperature-dependence of such a solid is governed by the single shift function  $\phi$ . Moreover, an arbitrary disposition over the temperature-dependence of the response parameters in (2.30) is not only incompatible with the temperature-time equivalence hypothesis but is also readily seen to be inadmissible on energetic grounds.

The implications of the equivalence postulate for elastic, Maxwell, and Kelvin solids were examined in [40]. As is at once apparent from (2.15), elastic materials with temperature dependent moduli do not belong to the class of thermo-rheologically simple viscoelastic solids. On the other hand, the shear response of a thermo-rheologically simple Maxwell solid is found to be characterized by

$$\dot{s}_{ij} + \frac{1}{\tau} s_{ij} = 2\mu \dot{e}_{ij}, \quad (5.13)$$

in which  $\mu$  and  $\tau$  are the shear modulus and the relaxation time of the solid; whereas  $\mu$  remains constant in the present instance,  $\tau$  obeys

$$\tau(T) = \frac{\tau_0}{\varphi(T)} (T_1 \leq T \leq T_2), \quad \tau_0 = \tau(T_0), \quad (5.14)$$

and is thus a monotone decreasing function of the temperature. Analogous conclusions apply to the dilatational response and to solids of the Kelvin type. It is interesting to note that the ad-hoc assumptions regarding the temperature-independence of certain response parameters made in [26] to [33] are in fact precise consequences of the temperature-time equivalence hypothesis. Before leaving the present subject we observe that, as pointed out by Lee and Rogers [41], the shortcomings of the differential operator law (i.e. of finite-spectrum models) are apt to be even more pronounced once the (thermo-rheologically simple) temperature dependence of the material is taken into account, in view of the concomitant contraction of the time scale.

We are now in a position to state the standard mixed boundary-value problem in the quasi-static linear theory of thermo-rheologically simple viscoelastic solids. A formulation based on the modified relaxation integral law may be phrased as follows. One is to find field histories  $u_1, \varepsilon_{1j}, \sigma_{1j}$  which — for given  $R, B_1, B_2$ , known  $G_1, G_2, \alpha, T_0, \varphi$ , and prescribed  $F_1, T, u_1^b, s_1^b$  — satisfy (2.1), (2.2), (2.3), (2.4), (5.5), (5.6) on  $\mathbb{R} \times (-\infty, \infty)$  and meet the initial conditions (2.8), as well as the boundary conditions (2.9). This problem suggests the subsequent generalization of the definition of a viscoelastic state introduced in Section 2.

Thermo-rheologically simple viscoelastic state. We say that  $[u_1, \varepsilon_{1j}, \sigma_{1j}]$  belongs to the class of thermo-rheologically simple viscoelastic states on  $R \times (-\infty, \infty)$  corresponding to the data  $G_1, G_2, \alpha, T_0, \varphi, F_1, T$  for the temperature range  $[T_1, T_2]$ , and write

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] \in \mathcal{J}[G_1, G_2, \alpha, T_0, \varphi, F_1, T] \text{ on } R \times (-\infty, \infty) \quad (5.15)$$

if:

- (a)  $G_1, G_2, u_1, \varepsilon_{1j}, \sigma_{1j}$ , and  $F_1$  meet requirements (a), (b) in the definition of a viscoelastic state;
- (b)  $\varphi$  is continuous on  $[T_1, T_2]$  and obeys (5.3);
- (c)  $T = T_0$  on  $R \times (-\infty, 0)$ , is continuous on  $R \times [0, \infty)$ , and has values in  $[T_1, T_2]$ ;
- (d) equations (2.1), (2.2), (2.3), (2.4), (5.5), (5.6) hold on  $R \times (-\infty, \infty)$ .

Evidently, (5.15) in conjunction with  $\varphi(T)=1$  for  $T$  in  $[T_1, T_2]$  implies (2.10), whence this specialization of the shift function  $\varphi$  yields a reduction of the present theory to the theory of temperature-independent materials treated in the preceding sections.

## 6. Implications of thermo-rheologically simple behavior, applications.

Our current objective is the compilation of a few general conclusions pertaining to the theory of thermo-rheologically simple viscoelastic solids. Most of the theorems about to be cited are extensions of, and upon the appropriate specialization

of  $\varphi$  reduce to, analogous propositions in Section 3. We state first the

Uniqueness theorem. Suppose

$$\left. \begin{aligned} [u_1, \varepsilon_{1j}, \sigma_{1j}] &\in \mathcal{J}[G_1, G_2, \alpha, T_0, \varphi, F_1, T] \text{ on } R \times (-\infty, \infty) \\ [u_1', \varepsilon_{1j}', \sigma_{1j}'] &\in \mathcal{J}[G_1, G_2, \alpha, T_0, \varphi, F_1, T] \text{ on } R \times (-\infty, \infty) \end{aligned} \right\} \quad (6.1)$$

and let

$$u_1 = u_1' \text{ on } B_1 \times (-\infty, \infty), \quad S_1 = S_1' \text{ on } B_2 \times (-\infty, \infty). \quad (6.2)$$

Then

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] = [u_1', \varepsilon_{1j}', \sigma_{1j}'] + [w_1, 0, 0] \text{ on } R \times (-\infty, \infty), \quad (6.3)$$

where  $w_1 = 0$  on  $R \times (-\infty, 0)$  and  $w_1$  represents an (infinitesimal) rigid motion of the entire body on  $R \times [0, \infty)$ .

This result is implied by a more inclusive uniqueness theorem established in [42], the latter being applicable also to ablating viscoelastic solids. It is interesting to observe that no restrictions on the shift function  $\varphi$ , beyond continuity and (5.3), are needed to assure uniqueness in the present circumstances. The next theorem may be inferred directly from the definitions of stationary elastic and thermo-rheologically simple viscoelastic states with the aid of (5.8), (5.9), and Theorem 1.2 in [1].

Initial response. Let  $[u_1, \varepsilon_{1j}, \sigma_{1j}]$  meet (5.15). Then

$$[u_1^0, \varepsilon_{1j}^0, \sigma_{1j}^0] \in \mathcal{L}[\mu, \kappa, \alpha, F_1^0, \Theta^0] \text{ on } R \quad (6.4)$$

with

$$\mu = \frac{1}{2} G_1^0, \quad \kappa = \frac{1}{3} G_2^0. \quad (6.5)$$

Consequently the initial state is again a stationary elastic state and may be found directly from the initial field and surface data. We turn now to a proposition which follows from a result cited<sup>23</sup> in Section 3.

Volume change. Let  $[u_1, \varepsilon_{1j}, \sigma_{1j}]$  meet (5.15) and suppose

$$G_2 = 3\kappa h \text{ on } (-\infty, \infty), \quad (6.6)$$

where  $\kappa$  is a constant (elastic dilatational response). Then the volume change is given by (3.17), (3.19).

The subsequent result was deduced in [6] and extends an earlier observation due to Hilton [9].

Stress-free temperature field histories. Assume (5.15) holds and suppose  $S_1=0$  on  $B \times [0, \infty)$ ,  $F_1=0$  on  $R \times [0, \infty)$ . Then  $\sigma_{1j}=0$  on  $R \times (-\infty, \infty)$  if and only if

$$\Theta(\underline{x}, t) = a_0(t) + a_1(t)x_1 \text{ for } (\underline{x}, t) \text{ in } R \times [0, \infty), \quad (6.7)$$

where  $a_0, a_1$  are functions continuous on  $[0, \infty)$ .

Related theorems concerning stress-free temperature fields, applicable to plane strain and generalized plane stress, are given in [6], where the two-dimensional theory of thermorheologically simple viscoelastic solids is explored in some detail. The two-dimensional considerations in [6] also led to a

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<sup>23</sup> See the remarks following (3.19).

connection between the plane-strain and the generalized plane-stress solution associated with one and the same plane problem.<sup>24</sup>

The chief analytical complications introduced by the departure from temperature-independent viscoelastic behavior arise in the actual solution of boundary-value problems. Unfortunately, the theory of integration presented in Section 4 does not admit a useful extension to thermo-rheologically simple viscoelastic media. This is true in particular of the powerful correspondence principle discussed at the end of Section 4, which yields a reduction of the viscoelasticity problem (for a solid with a temperature-independent response) to a problem in steady-state thermoelasticity theory.

To bring the difficulty to which we have just alluded into focus, we observe first that the integrals appearing in the modified relaxation law (5.5) are no longer of the convolution type. Consequently, an application of the Laplace transform (with respect to the physical time) to (5.5) fails to furnish algebraic stress-strain relations in the transform domain. The alternative version (5.9) of the modified relaxation law, which has the requisite convolution structure, suggests the possibility of referring also the remaining field equations, as well as the boundary conditions, to the new independent variables  $(\underline{x}, \xi)$  with a view toward a subsequent elimination of the reduced time by means of the Laplace transform. Such a procedure in general does not result in a worthwhile simplification of the problem. Indeed,

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<sup>24</sup> For the analogous connection in two-dimensional thermoelasticity theory, see Mindlin and Salvadori [43], p.762.

let us use (5.6), (5.8) to effect the proposed change of variables and let us agree to write  $\hat{f}_{,1}$  and  $\hat{f}'$  for the space derivatives and for the reduced-time derivative of a function  $\hat{f}$  of position and of the reduced time. Then the displacement-strain relations (2.1) become

$$\hat{\varepsilon}_{1j} = \frac{1}{2}(\hat{u}_{1,j} + \hat{u}_{j,1}) + \frac{1}{2}(\hat{u}_{1\rho,j} + \hat{u}_{j\rho,1}), \quad (6.8)$$

whereas the stress equations of equilibrium (2.2) pass over into

$$\hat{\sigma}_{1j,j} + \hat{\sigma}_{1j\rho,j} + \hat{F}_1 = 0, \quad \hat{\sigma}_{j1} = \hat{\sigma}_{1j}. \quad (6.9)$$

Because of the terms involving  $\rho_{,1}$ , the transforms of (6.8), (6.9) (taken with respect to the reduced time) no longer possess the desired structure of (2.1), (2.2), unless  $\rho_{,1}$  vanishes. Such will be the case, as is clear from (5.6), if T is a function of the time alone. In this special event one is accordingly led to a significant generalization of the correspondence principle. To avoid unduly cumbersome notation in the subsequent statement of this extension we shall write

$$\tilde{f}(\underline{x}, \eta) = \int_0^\infty \hat{f}(\underline{x}, \xi) \exp(-\eta\xi) d\xi \quad (6.10)$$

for the Laplace transform with respect to the reduced time of a function  $\hat{f}$  with values  $\hat{f}(\underline{x}, \xi)$ .

Correspondence principle for purely time-dependent temperature histories. Suppose

$$[u_1, \varepsilon_{1j}, \sigma_{1j}] \in \mathcal{J}[G_1, G_2, \alpha, T_0, \varphi, F_1, T] \text{ on } R \times (-\infty, \infty), \quad (6.11)$$

with T a function of the time alone, and let

$$u_1 = u_1^b \text{ on } B_1 \times (-\infty, \infty), \quad S_1 = S_1^b \text{ on } B_2 \times (-\infty, \infty). \quad (6.12)$$

Assume that  $G_\beta(\xi)$  ( $\beta=1,2$ ),  $\hat{u}_1(\underline{x},\xi)$ ,  $\hat{\varepsilon}_{1j}(\underline{x},\xi)$ ,  $\hat{F}_1(\underline{x},\xi)$ , and  $\hat{T}(\xi)$ , for every x in R, are all of the exponential order  $O(\exp(s_0\xi))$  as  $\xi \rightarrow \infty$ , where  $s_0$  is a (real) constant. Then, for each  $\eta$  with  $\text{Re}(\eta) > s_0$ ,

$$[\tilde{u}_1(\cdot, \eta), \tilde{\varepsilon}_{1j}(\cdot, \eta), \tilde{\sigma}_{1j}(\cdot, \eta)] \in \mathcal{L}[\mu(\eta), \kappa(\eta), \alpha, \tilde{F}_1(\cdot, \eta), \tilde{\Theta}(\eta)] \text{ on } R, \quad (6.13)$$

where

$$\mu(\eta) = \frac{1}{2} \eta \tilde{G}_1(\eta), \quad \kappa(\eta) = \frac{1}{3} \eta \tilde{G}_2(\eta), \quad (6.14)$$

and further

$$\tilde{u}_1(\cdot, \eta) = \tilde{u}_1^b(\cdot, \eta) \text{ on } B_1, \quad \tilde{S}_1(\cdot, \eta) = \tilde{S}_1^b(\cdot, \eta) \text{ on } B_2. \quad (6.15)$$

The preceding analogy becomes trivial if  $B=B_2$ ,  $F_1=0$  on  $R \times (-\infty, \infty)$ , and  $S_1=0$  on  $B \times (-\infty, \infty)$ . In this instance the analogy merely confirms our previous conclusion<sup>25</sup> that  $\sigma_{1j}=0$  on  $R \times (-\infty, \infty)$ , in view of the well-known theorem on stress-free temperature fields in thermoelasticity theory ([3], Art. 3.9).

Suppose next that for non-negative time the temperature  $T$ , is a function of position alone. In this second degenerate case the modified relaxation integral law (5.5) may once again be written in terms of convolutions with respect to the physical time, as is apparent from (5.6). In fact one now has

<sup>25</sup> Note that  $\Theta$  conforms to (6.7) since  $T$  is at present independent of position.

$$s_{ij} = H_1 * d e_{ij}, \quad \sigma_{kk} = H_2 * d (\epsilon_{kk} - 3\alpha\Theta), \quad (6.16)$$

provided

$$H_p(\underline{x}, t) = G_p(t\varphi(T(\underline{x}))) \text{ for } (\underline{x}, t) \text{ in } R \times [0, \infty). \quad (6.17)$$

Upon taking the Laplace transform of (6.16) one finds that

$$\left. \begin{aligned} \bar{s}_{ij}(\underline{x}, \eta) &= 2\mu(\underline{x}, \eta) \bar{e}_{ij}(\underline{x}, \eta), \\ \bar{\sigma}_{kk}(\underline{x}, \eta) &= 3\kappa(\underline{x}, \eta) [\bar{\epsilon}_{kk}(\underline{x}, \eta) - 3\alpha\bar{\Theta}(\underline{x}, \eta)] \end{aligned} \right\} \quad (6.18)$$

for each fixed  $\eta$  with a sufficiently large real part and for all  $\underline{x}$  in  $R$ , if

$$\mu(\underline{x}, \eta) = \frac{1}{2} \eta \bar{H}_1(\underline{x}, \eta), \quad \kappa(\underline{x}, \eta) = \frac{1}{3} \eta \bar{H}_2(\underline{x}, \eta). \quad (6.19)$$

But (6.18) are the stress-strain relations appropriate to an inhomogeneous linear elastic solid. Consequently, if the temperature depends on position alone, the original boundary-value problem in thermo-viscoelasticity theory is reducible to a steady-state thermoelastic problem for an inhomogeneous medium. Unhappily, this generalization of the correspondence principle is of very limited practical interest since the complexity of the reduced problem is apt to be comparable to that of the original one.

For the differential operator law an analogous extension of the correspondence principle to temperature-dependent materials was given earlier by Hilton and Russel [44] in a paper which is based on their previous report [20] with Hassan. The analysis in [44] is confined to differential operator laws with temperature-dependent material parameters and thus presupposes a medium with

a finite relaxation or retardation spectrum. The analogy deduced in [44] for purely time-dependent temperature distributions is limited to mechanically incompressible solids (although thermal expansion is included) and is an extension of Alfrey's [15] principle.<sup>26</sup> On the other hand, no restriction is placed in [44] on the linear viscoelastic volumetric response for the case of a purely position-dependent temperature field; the correspondence principle obtained in this instance is an elementary generalization of Read's [17] work in the isothermal theory. Finally, [44] contains a sketch of an approximative approach to the general case of a position and time-dependent temperature field history, based on the assumption that the material may be regarded as stepwise temperature-independent (in time). The extent to which such an approximation scheme is computationally feasible remains to be assessed.

We turn now to available applications of the quasi-static linear theory of thermo-rheologically simple viscoelastic solids. As was mentioned before, all but one of the special investigations listed at the beginning of Section 5 are based on temperature-dependent viscoelastic models that fall within this category, although the problems treated were not intentionally selected to illustrate the general theory under present consideration.

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<sup>26</sup> In connection with this extension of Alfrey's scheme the authors remark that the effect of body forces and surface tractions may be determined separately from, and may be superposed upon, the ensuing thermal stresses and deformations. Such a superposition is in fact not permissible for a temperature-dependent material. Indeed, as has been pointed out already, a purely time-dependent temperature field would, in the absence of loads, always induce vanishing thermal stresses.

The temperature-time equivalence hypothesis was applied by Morland and Lee [35] to the plane-strain analysis of an incompressible hollow circular cylinder which is exposed to a stationary radial distribution of temperature and is subjected to a suddenly applied uniform internal pressure.

Implications of the theory in circumstances that involve temperature distributions depending on both position and time were studied in [40], which contains exact solutions to two space problems of this type. The first of these concerns an infinite slab which, in the absence of loads, is confined to a purely transverse motion, the stress-inducing temperature field history being permitted to vary arbitrarily with the thickness coordinate and the time. The second problem aims at the thermal stresses and deformations generated in a sphere by an arbitrary transient radial temperature distribution. Both solutions accommodate an unrestricted (thermo-rheologically simple) temperature dependence of the material and apply to solids with a continuous relaxation spectrum, except that in the second problem the volumetric response is assumed to be elastic.<sup>27</sup>

We return here briefly to the example of the slab treated in [40] since certain aspects of this problem have a

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<sup>27</sup> Although this assumption, which is common in the stress analysis of viscoelastic solids, is in general agreement with experimental findings, the available quantitative information regarding volumetric viscosity effects appears to be inadequate.

wider significance. Thus consider an infinite plate of constant thickness  $2a$  and choose the coordinate frame in such a way that  $x_3=0$  coincides with the middle plane of the plate. Then  $R$  is the region characterized by  $-a \leq x_3 \leq a$  and  $B$  consists of the pair of bounding planes  $x_3=\pm a$ . We now seek a thermo-rheologically simple viscoelastic state<sup>28</sup>  $[u_1, \epsilon_{1j}, \sigma_{1j}]$  on  $R \times (-\infty, \infty)$  corresponding to the data  $G_1, G_2, \alpha, T_0, \varphi, F_1, T$ , subject to the boundary conditions

$$\sigma_{31} = 0 \quad \text{on } B \times (-\infty, \infty). \quad (6.20)$$

We assume further that  $T, u_3$  are functions of  $(x_3, t)$  alone and

$$F_1 = u_1 = u_2 = 0 \quad \text{on } R \times (-\infty, \infty). \quad (6.21)$$

The foregoing constraining assumptions are consistent with the governing field equations and boundary conditions, which at present are readily found to imply<sup>29</sup> that all field histories depend solely on  $(x_3, t)$ , while

$$\epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{1j} = 0 \quad (1 \neq j) \quad \text{on } R \times (-\infty, \infty), \quad (6.22)$$

$$\sigma_{11} = \sigma_{22}, \quad \sigma_{33} = 0, \quad \sigma_{1j} = 0 \quad (1 \neq j) \quad \text{on } R \times (-\infty, \infty), \quad (6.23)$$

whereas

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<sup>28</sup> Note that, contrary to our previous assumption,  $R$  is not bounded in this instance.

<sup>29</sup> See [40] for details.

$$\hat{\sigma} = -\hat{\varepsilon} * dG_1, \quad 2\hat{\sigma} = (\hat{\varepsilon} - 3\alpha\hat{\Theta}) * dG_2 \quad \text{on } \mathbb{R} \times (-\infty, \infty), \quad (6.24)$$

if we conveniently write  $\sigma$  and  $\varepsilon$  for  $\sigma_{11}$  and  $\varepsilon_{33}$ , respectively.

The elimination of  $\hat{\varepsilon}$  among the pair of integral equations (6.24), in view of the algebra of Stieltjes convolutions (Section 1 in [1]), now leads to

$$\hat{\sigma} = -3\alpha A * d\hat{\Theta} \quad \text{on } \mathbb{R} \times (-\infty, \infty), \quad (6.25)$$

where  $A$  is the auxiliary response function defined by

$$A = G_1 * dG_2 * d(2G_1 + G_2)^{-1} \quad \text{on } (-\infty, \infty). \quad (6.26)^{30}$$

Bearing in mind (2.7), (5.6), (5.7), (5.8), and putting  $x_3 = x$ , we draw from (6.25) that the desired stress  $\sigma = \sigma_{11} = \sigma_{22}$  admits the integral representation

$$\sigma(x, t) = -3\alpha \int_{t'=-\infty}^t A(\xi - \xi') d\Theta(x, t') \quad \text{for } (x, t) \text{ in } \mathbb{R} \times (-\infty, \infty), \quad (6.27)$$

with

$$\xi = \rho(x, t) = \int_0^t \varphi(T(x, t')) dt', \quad \xi' = \rho(x, t'). \quad (6.28)$$

Moreover, (6.26), by virtue of Theorems 1.2, 1.3 in [1], for  $t$  in  $[0, \infty)$  is equivalent to

$${}^0GA(t) + \int_0^t \dot{G}(t-t') A(t') dt' = L(t), \quad (6.29)^{31}$$

where

<sup>30</sup> The same response function is also found to play a crucial role in the problem of the sphere [40].

<sup>31</sup> Recall (2.13).

$$G(t) = 2Q_1(t) + Q_2(t), \quad L(t) = \overset{0}{Q}_2 Q_1(t) + \int_0^t Q_1(t-t') \dot{Q}_2(t') dt', \quad (6.30)$$

so that  $A$  satisfies a linear integral equation of Volterra's second type. Finally, if the requisite Laplace transforms exist, (6.29) and (6.30) furnish

$$\bar{A}(\eta) = \frac{\bar{Q}_1(\eta)\bar{Q}_2(\eta)}{2\bar{Q}_1(\eta) + \bar{Q}_2(\eta)}. \quad (6.31)$$

Formula (6.27) was obtained in [40] by applying the Laplace transform to the pair of integral equations (6.24) and by subsequently inverting the solution of the resulting pair of algebraic equations. As was noted later by Lee and Rogers [41] and is clear from the preceding derivation, there is no need to involve the transform calculus. A similar observation is relevant to the numerical evaluation of the solution under discussion, which was carried out in [40] for a commercial polymethyl methacrylate on the basis of available relaxation data and test results for the shift function  $\phi$ . In this connection the auxiliary response function  $A$  was determined from (6.31). The laborious procedure adopted in [40] consisted in first devising a convenient analytical approximation to the relaxation data that permitted the analytical computation of  $\bar{A}$ ; the required values of  $A$  were then found with the aid of two asymptotic inversions — one applicable to large, the other to small values of the time. In contrast, Lee and Rogers [41] subsequently re-computed  $A$  by solving the integral equation (6.29) directly on an electronic computer. Their

procedure, which stays closer to the actually measured data, is evidently more direct and also more accurate.

The values of  $A$  obtained by the two alternative methods described above are compared in Figure 1, in which  $E$  is Young's modulus of the initial elastic response. Figure 2, which is taken from [40], displays the time-dependence of the normal stress  $\sigma$  in the middle plane of the slab for polymethyl methacrylate. In the underlying computations it was assumed that  $a^2/k = 1$  hour,  $k$  being the thermal diffusivity of the material; this choice corresponds to a slab thickness  $2a$  of approximately 5.7 cm. The temperature field history used in these calculations is that supplied by the solution to an elementary transient heat-conduction problem: the entire slab is initially at the uniform base temperature  $T_0 = 80^\circ\text{C}$  and its faces at  $t=0$  are suddenly raised to the temperature  $T_1=110^\circ\text{C}$ , which is steadily maintained thereafter.

Curve 1 in Figure 2 shows the stress values obtained from (6.27). Curve 2 exhibits the behavior predicted if the temperature-dependence of the response is disregarded and the analysis is based on relaxation data appropriate to the average surface temperature of  $95^\circ\text{C}$ . Curve 3, finally, shows the results obtained if one neglects all viscosity effects as well and determines  $\sigma$  on the supposition of purely elastic behavior, using the elastic constants characteristic of the initial viscoelastic response. As is apparent, the three curves almost coalesce for an initial period of about 10 minutes, during which the "actual"

response is nearly elastic and hence temperature-independent. For approximately the first 20 minutes Curve 2 slightly underestimates the stress values predicted by Curve 1 since during this stage the temperature of the middle plane is less than the average temperature of  $95^{\circ}\text{C}$  and consequently the actual relaxation process is slower than that underlying Curve 2. Later on, however, Curve 2 progressively overestimates the stress values depicted by Curve 1 until, after a two-hour period, the actual stress is less than 15 per cent of the corresponding value in Curve 2. These results demonstrate once again the unrealistic character of any viscoelastic thermal-stress analysis that leaves out of account the temperature-dependence of the material properties.

The relative merits of the direct numerical integration of integral equations over the use of integral transform techniques in viscoelastic stress analysis were further discussed and illustrated by Lee and Rogers in [45]. Apart from the fact that this approach makes more efficient use of experimental data, it obviates the need for long-time extrapolations from the data beyond the time range which the desired solution of the problem is to cover. Further, such a treatment extends the scope of the analysis to problems which are not amenable to a Laplace-transform solution. Among these are mixed problems in which the surface tractions and displacements are prescribed over time-dependent subsets of the boundary and problems in which the boundary itself is a function of the time (surface ablation).

An example of the latter type is dealt with numerically in [45], where the solution to the problem of the sphere obtained in [40] is generalized to accommodate an ablating spherical body. In addition, [45] contains a formal solution to the problem of an ablating spherical shell; for fixed boundaries this problem was also treated previously in [6].

Although the developments just described are promising, it is well to keep in mind that boundary-value problems within the theory under consideration are reducible to the solution of independent integral equations only in highly exceptional circumstances. Furthermore, all of the applications available to date depend on the fortuitous possibility that the required space and time integrations can be carried out separately. While certain less degenerate problems may nowadays be accessible to a purely numerical treatment, such prospects should not detract from the need for a systematic theory of integration.

## 7. Concluding remarks.

Throughout the preceding developments the material has been considered to be isotropic with regard to both its mechanical and its thermal response. For an anisotropic linear viscoelastic material the constitutive relations (2.6) give way to

$$\sigma_{ij} = (\epsilon_{ij} - \alpha_{ij}\Theta) * dG_{ijkl}, \quad (7.1)$$

in which  $G_{ijkl}$  and  $\alpha_{ij}$  are the components of the tensor of relaxation functions and of the thermal-expansion tensor, respectively. Furthermore,  $\alpha_{ji} = \alpha_{ij}$  and

$$G_{ijkl} = G_{jikl} = G_{ijlk}. \quad (7.2)$$

The first of (7.2) follows from the symmetry of the stress tensor, while the second entails no loss in generality because of the symmetry of the strain tensor. In addition, it is usually assumed that

$$G_{ijkl} = G_{klij}. \quad (7.3)$$

This last set of symmetry relations, which is implied by (7.2) in the special case of isotropy, expresses an independent requirement. The available theoretical support for the validity of (7.3) depends on thermodynamic arguments involving an appeal to Onsager's reciprocity relations.<sup>32</sup>

If (7.3) are presumed to hold, the generalization to homogeneous anisotropic solids of most of the theoretical results discussed in this paper, presents no difficulties. This is true, in particular, of the correspondence principle discussed in Section 4 and Section 6, which now leads to a tie between the linear theories of anisotropic viscoelastic and elastic solids, as suggested by Biot [49]. An extension of this principle to inhomogeneous (isotropic or anisotropic) viscoelastic solids is equally elementary,<sup>33</sup> though of little practical consequence.

<sup>32</sup> See, for example, Biot [46],[47]. A discussion of this issue may be found in a recent note by Rogers and Pipkin [48], as well as in [5]. Note that the counterpart of (7.3) in elasticity theory is implied by the existence of an elastic potential.

<sup>33</sup> In this connection see [20],[44], as well as Hilton and Dong [50].

Next, we have so far considered only the quasi-static analysis of thermal stresses in viscoelastic solids. Thermo-viscoelastic inertia effects have been the object of several recent special investigations. Among these we cite publications by Katasonov [51], Nowacki [23],[52],[53],[54], and Zorawski [55], [56],[57],[58]. The physical significance of these investigations is difficult to assess since they invariably presuppose a temperature-independent material, do not include quantitative discussions of the results obtained, and usually refer only to instantaneous temperature changes.<sup>34</sup>

Further, nothing has been said as yet about thermo-mechanical coupling effects, which have been consistently disregarded. An interesting study of such effects is due to Hunter [60], whose analysis is set within the framework of the theory of thermo-rheologically simple viscoelastic solids. As suggested by the unassuming title of [60], this subject is in need of further consideration.

Finally, it may be well to emphasize once more that the present paper is confined essentially to linear thermal stress analysis for viscoelastic solids. Thus no attention has been given to the influence of nonlinear viscosity effects and to finite deformations, which haunt the realistic treatment of thermal stresses in metals at elevated temperatures.

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<sup>34</sup> Related studies in thermoelasticity theory (see, for example, [59]), indicate that the size of such inertia effects is drastically reduced once the fiction of sudden temperature changes is abandoned.

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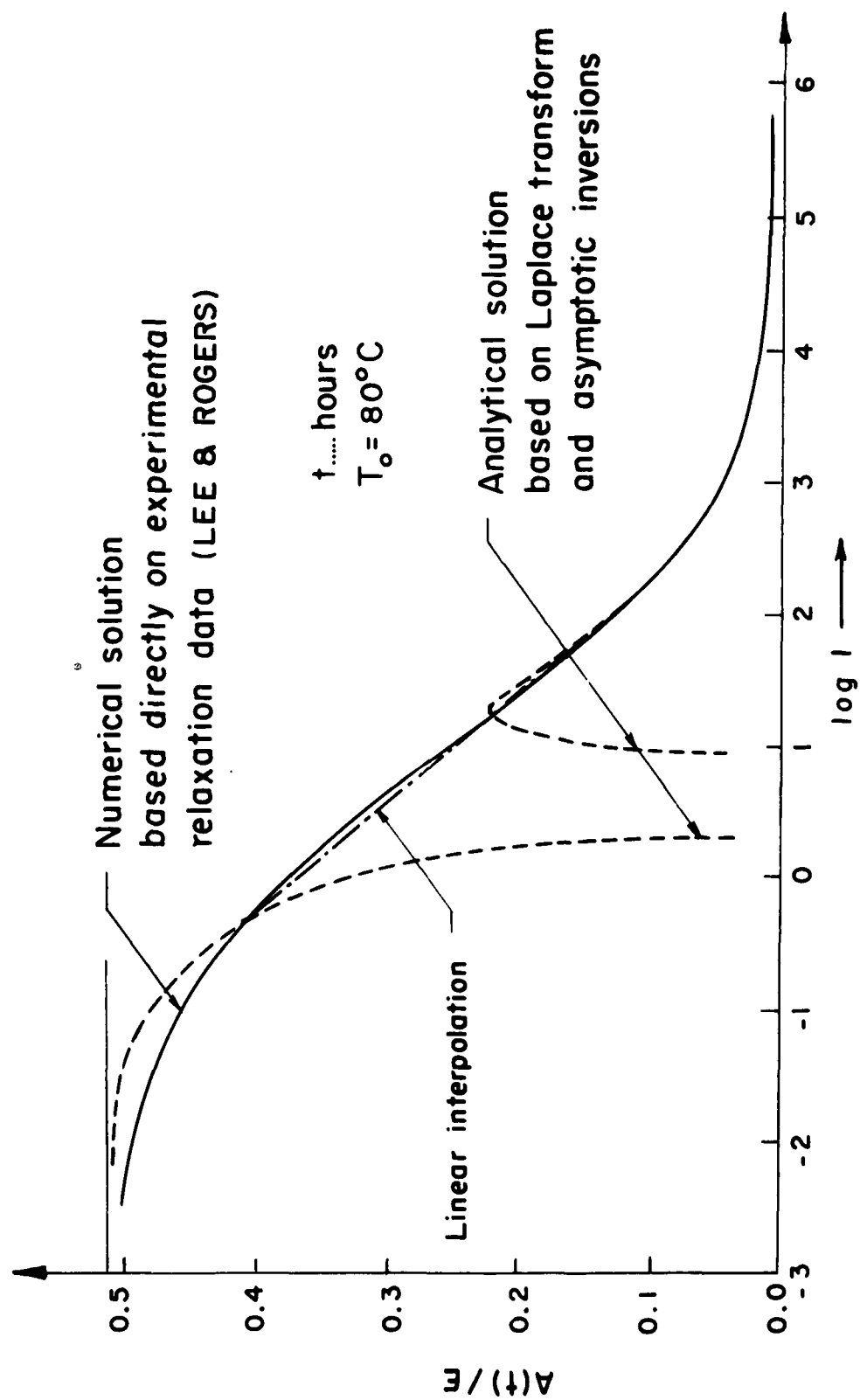
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Fig.1  $A(t)$  for polymethyl methacrylate

- ① Temperature-dependent response:  $T_0 = 80^\circ\text{C}$ ,  $T_1 = 110^\circ\text{C}$
- ② Temperature-independent response based on behavior at  $95^\circ\text{C}$
- ③ Elastic response based on initial behavior

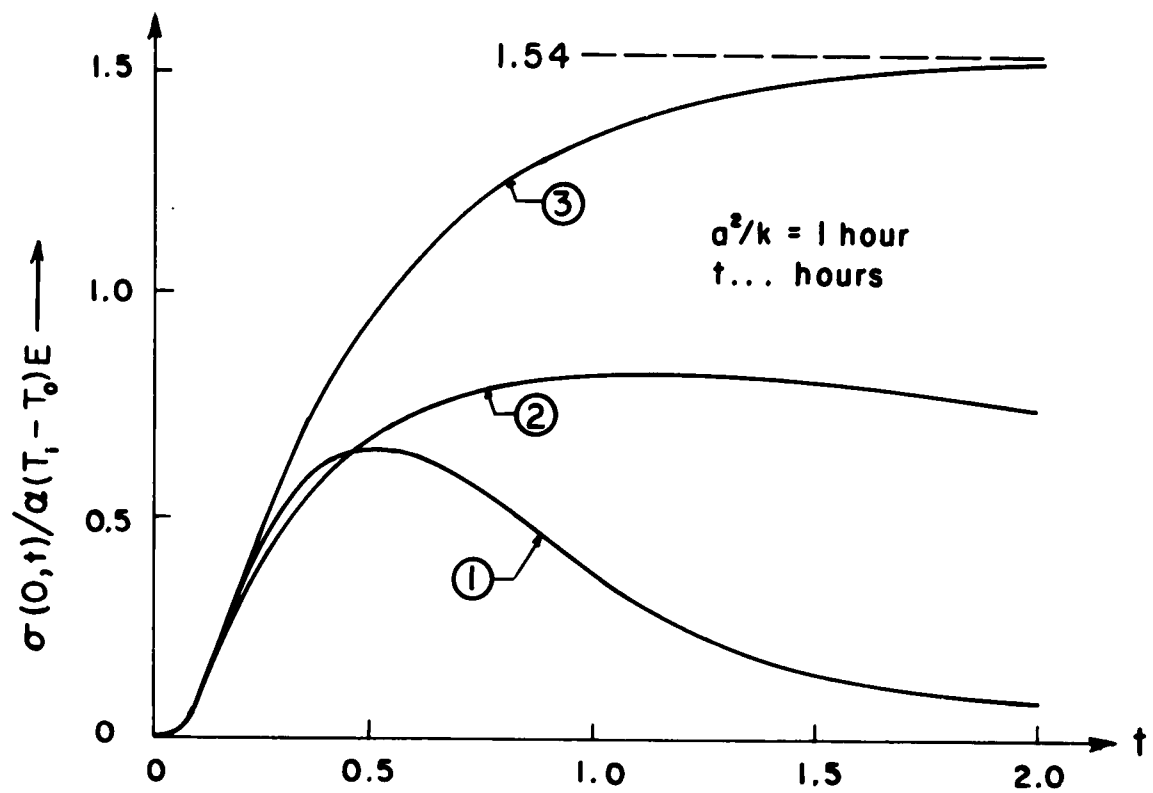


Fig.2 Slab problem. Time-dependence of  $\sigma$  in middle plane for polymethyl methacrylate